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LETTERS TO THE EDITOR

EFFECTS OF MODE INCOMPLETENESS ON THE QUALITY OF VARIOUS MODEL-UPDATING ALGORITHMS

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1. INTRODUCTION

Sophisticated analytical models are often needed to analyze and predict the dynamical behavior of complete structures. With the proliferation of digital computers, new methods of analysis have been developed, especially in the method of finite elements. Once the finite element model of a physical system is constructed, its validity is often checked by comparing its analytical modes of vibration with those obtained from a modal survey. If the modal survey and the analytical predictions are in subjective agreement, then the analytical model can be used with confidence for future analysis. If the correlation between the two is poor, then assuming the experimental measurements to be correct, the analytical model must be adjusted so that the agreement between the analytical predictions and the test results is improved. The updated model may then be considered a better representation of the physical structure than the initial analytical model, and it can then be used with reasonable accuracy to assess the stability and control characteristics and to predict the dynamical responses of the structure. The above process of correcting the system matrices is known as *model updating*.

In recent years, many methods have been developed to improve the quality of the analytical finite element models using test data. Detailed discussion of every approach is beyond the scope of this note, and interested readers are referred to the survey papers by Mottershead and Friswell [1] and Imregun and Visser [2]. In a recent paper [3], new approaches are developed that use measured natural frequencies and mode shapes to update the analytical mass and stiffness matrices of a structure. Using the measured modes of vibration of a mass-modified structure and the initial modal survey, the mass matrix of the system can be corrected, after which the stiffness matrix can be updated by requiring it to satisfy the generalized eigenvalue problem associated with the free vibration of the structure. Manipulating the unknown system matrices into vector forms, the well-known and readily available connectivity information can be enforced to preserve the physical configuration of the structure and to reduce the computational efforts required to correct the system matrices.

In this technical note, detailed numerical experiments will be performed to compare the results of the mass and stiffness updating algorithms introduced in reference [3] against the Lagrange multipliers updating formalisms [4, 5] and the perturbation updating approach [6]. In addition, error parameters will be introduced to gauge the quality of the updates. The effects of mode incompleteness on these error parameters will also be investigated. Finally, heuristic criterion regarding the minimum number of measured modes needed to ensure a sufficiently accurate updated mass or stiffness matrix will be established.

2. RESULTS

The Lagrange multipliers approaches [4, 5], the perturbation scheme [6] and the proposed model updating algorithm [3] are applied to the simple system of Figure 1, whose mass matrix is diagonal and whose stiffness matrix is symmetric and tri-diagonal. The analytical masses and stiffnesses are 2·000 kg and 5·000 N/m, respectively. The actual masses and stiffnesses are given in Table 1, for $N = 25$.

Experimentally, it is nearly impossible to measure the same number of modes as the number of degrees of freedom of the analytical model. Thus, the measured data are said to be incomplete. A problem unrelated to that previously described, but also commonly referred to as "incomplete" occurs when the measured eigenvector contains fewer

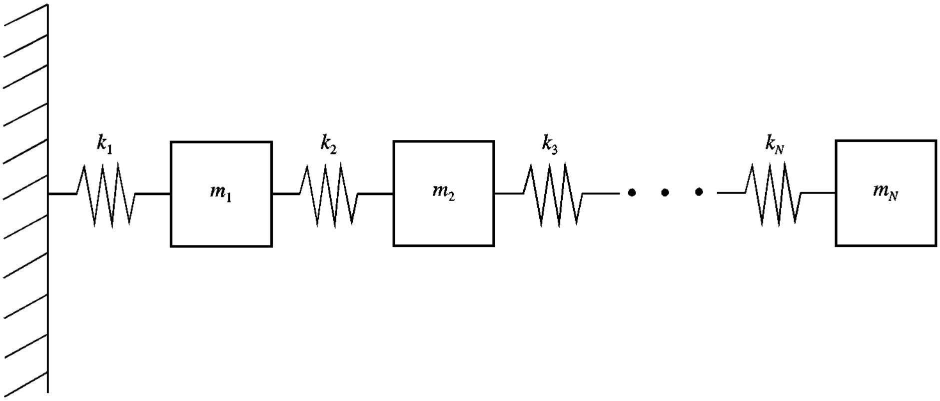

Figure 1. Simple chain of coupled oscillators.

TABLE 1

The actual masses (in kg) and stiffnesses (in N/m), for $N = 25$. The analytical masses and stiffnesses are $m_0 = 2{\cdot}000$ kg and $k_0 = 5{\cdot}000$ N/m, respectively

$\mathbf{m}_{actual}$	$\mathbf{m}_{actual}$	$\mathbf{k}_{actual}$	$\mathbf{k}_{actual}$
$m_1 = 1{\cdot}2942$	$m_{14} = 2{\cdot}6722$	$k_1 = 4{\cdot}1400$	$k_{14} = 6{\cdot}9877$
$m_2 = 2{\cdot}1831$	$m_{15} = 1{\cdot}3355$	$k_2 = 0{\cdot}8802$	$k_{15} = 5{\cdot}4973$
$m_3 = 1{\cdot}3117$	$m_{16} = 2{\cdot}6680$	$k_3 = 5{\cdot}6052$	$k_{16} = 5{\cdot}9483$
$m_4 = 2{\cdot}6581$	$m_{17} = 1{\cdot}3899$	$k_4 = 6{\cdot}5108$	$k_{17} = 5{\cdot}0320$
$m_5 = 2{\cdot}4371$	$m_{18} = 1{\cdot}8632$	$k_5 = 2{\cdot}9343$	$k_{18} = 6{\cdot}3608$
$m_6 = 1{\cdot}7651$	$m_{19} = 1{\cdot}6231$	$k_6 = 7{\cdot}1326$	$k_{19} = 6{\cdot}2726$
$m_7 = 2{\cdot}8502$	$m_{20} = 1{\cdot}1578$	$k_7 = 3{\cdot}3072$	$k_{20} = 7{\cdot}0572$
$m_8 = 1{\cdot}7984$	$m_{21} = 1{\cdot}1439$	$k_8 = 3{\cdot}2986$	$k_{21} = 6{\cdot}6026$
$m_9 = 1{\cdot}7793$	$m_{22} = 1{\cdot}8189$	$k_9 = 6{\cdot}2021$	$k_{22} = 4{\cdot}9326$
$m_{10} = 2{\cdot}7588$	$m_{23} = 1{\cdot}2320$	$k_{10} = 6{\cdot}6399$	$k_{23} = 6{\cdot}3932$
$m_{11} = 2{\cdot}1221$	$m_{24} = 1{\cdot}6389$	$k_{11} = 5{\cdot}9489$	$k_{24} = 5{\cdot}8004$
$m_{12} = 1{\cdot}1613$	$m_{25} = 2{\cdot}2254$	$k_{12} = 6{\cdot}3203$	$k_{25} = 6{\cdot}5935$
$m_{13} = 2{\cdot}0234$	—	$k_{13} = 3{\cdot}3357$	—

co-ordinates than are available from the analytical model. In this case, the measured eigenvectors must first be expanded before the proposed algorithms can be applied. Various mode expansion techniques can be found in references [2, 7]. In this note, it will be assumed that all the co-ordinates of the eigenvectors can be measured, and we will reserve the word "incomplete" to mean when the test measurements contain fewer modes than those of the analytical model. In practice, because the number of measured modes, N_e, needed to perform the update is almost always less than degrees of freedom, N, of the analytical model, of considerable interest is the effect of N_e on the quality of the updated systems matrices.

Theoretically, N_e may include any modes of vibration of the system. However, because the lower modes are typically easier to measure experimentally than the higher ones, the parameter N_e will be used in the subsequent analysis to represent the lowest N_e measured modes of the structure. Thus, when $N_e = 5$, the first five measured modes will be used to perform the update.

To assess the accuracy of the mass updating algorithm as a function of N_e, the following relative error parameter for the updated masses is introduced:

$$\varepsilon_m = \frac{|\mathbf{m}_{update} - \mathbf{m}_{actual}|}{|\mathbf{m}_{actual}|}, \tag{1}$$

where $\mathbf{m}_{update}$ and $\mathbf{m}_{actual}$ are vectors of length N whose elements are the updated and the actual masses, respectively, and $|\mathbf{a}|$ represents the Euclidean norm of the vector $\mathbf{a}$. In order to compare the improvement of the updated masses over their initial analytical values, a relative error parameter for the analytical masses is also introduced:

$$(\varepsilon_m)_0 = \frac{|\mathbf{m}_{analytical} - \mathbf{m}_{actual}|}{|\mathbf{m}_{actual}|}. \tag{2}$$

Similar expressions can be defined for the relative error parameters for the updated and analytical stiffnesses, denoted by ε_k and $(\varepsilon_k)_0$, respectively.

Once the structural parameters of the system have been corrected, the eigenvalues of the updated system are compared with those obtained during a modal survey to judge the accuracy of the updating algorithms. Relative error parameters for the eigenvalues (the natural frequencies squared) can be similarly defined as those in equations (1) and (2). For an updated model to be judged better than the initial analytical model, we must have $\varepsilon_m < (\varepsilon_m)_0$, $\varepsilon_k < (\varepsilon_k)_0$ and $\varepsilon_\lambda < (\varepsilon_\lambda)_0$. For an updated model to be considered accurate, ε_m, ε_k and ε_λ must be sufficiently small. Additionally, the smaller the error parameters are, the better the updated model is.

To execute the proposed mass updating algorithm, lumped masses of magnitude 0·2 kg are added to masses 1 and 25, which correspond to the two ends of the structure of Figure 1. Figure 2 shows the variations of ε_m, obtained by using the proposed mass updating algorithm, the Lagrange multipliers scheme and the perturbational method, as a function of N_e. Also shown is the corresponding $(\varepsilon_m)_0$, which is independent of N_e and is given by the horizontal line. Note the improvement in the updated mass parameters as N_e increases, for both the proposed and the Lagrange multipliers schemes. The proposed method returns updated masses that are increasingly more accurate than the Lagrange multipliers approach as N_e increases. For these two approaches, the experimental results are consistent with physical intution: the more information that is gathered about the physical system, the

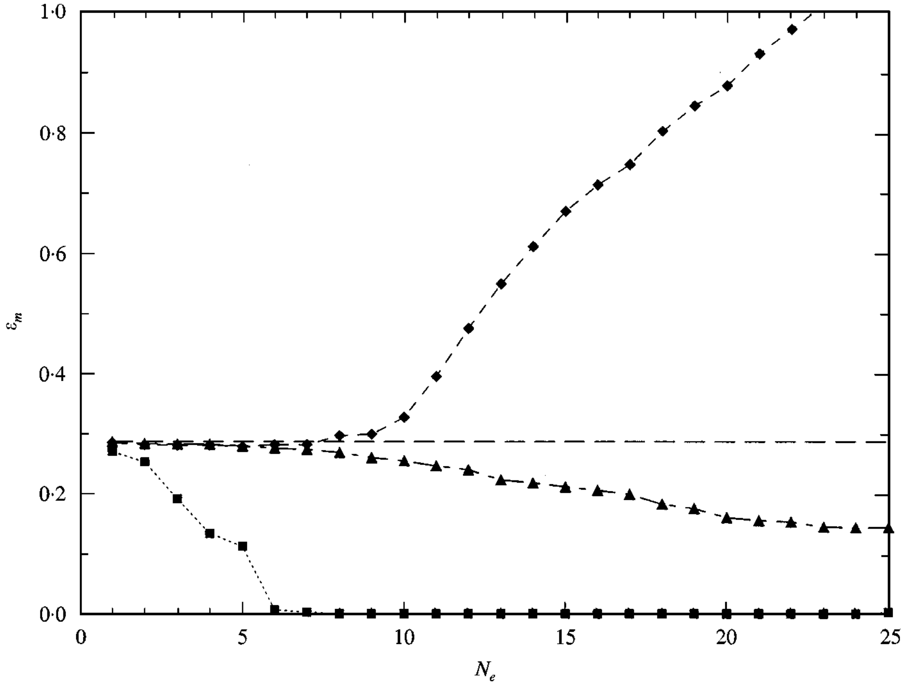

Figure 2. The mass error parameter, ε_m, obtained by using the proposed mass updating algorithm (■), the Lagrange multipliers scheme (▲), and the perturbation method (◆), as a function of the number of measured modes, N_e. The dotted horizontal line represents the mass error parameter of the analytical model, $(\varepsilon_m)_0$.

better the updated model becomes. The perturbation approach, on the other hand, returns updated masses that deviate considerably from those of the physical system for large N_e. In fact, for $N_e \geqslant 8$, the perturbation scheme returns adjusted masses that are *worst* than the initial analytical values.

The ε_m curve for the proposed mass-updating scheme (see Figure 2) also reveals the minimum number of measured modes that are needed in order to achieve a certain level of accuracy. From numerical experiments it was observed that more accurate solutions are obtained when the least-squares system becomes overdetermined. Because updating the mass matrix of Figure 1 involves solving a least-squares problem of size $N_e^2 \times N$ (see reference [3] for detailed discussion), to render the problem overdetermined requires $N_e \geqslant \sqrt{N}$ measured modes. Thus, for $N = 25$, at least five measured modes are needed to update the masses to ensure sufficient accuracy. Because the criterion regarding the minimum N_e needed to perform the mass update is formulated empirically, for the mass parameters of Table 1, accurate updated mass matrix is obtained for $Ne \geqslant 6$. The numerical results of Figure 2 also indicate that there is a saturation point beyond which additional information does not lead to any significant improvement in the corrected mass matrix.

Once the masses have been adjusted, the analytical stiffness matrix can be corrected by requiring it to satisfy the generalized eigenvalue problem that governs the free vibration of the system. Figure 3 shows the variations of ε_k, obtained by the Lagrange multipliers formalism, the perturbation approach, and the proposed updating scheme, as a function of N_e. Also shown is the corresponding $(\varepsilon_k)_0$, which does not vary with N_e and is given by the horizontal line. Note that ε_k decreases as N_e increases for the proposed stiffness updating algorithm. The observed results imply that the larger the knowledge space that is known

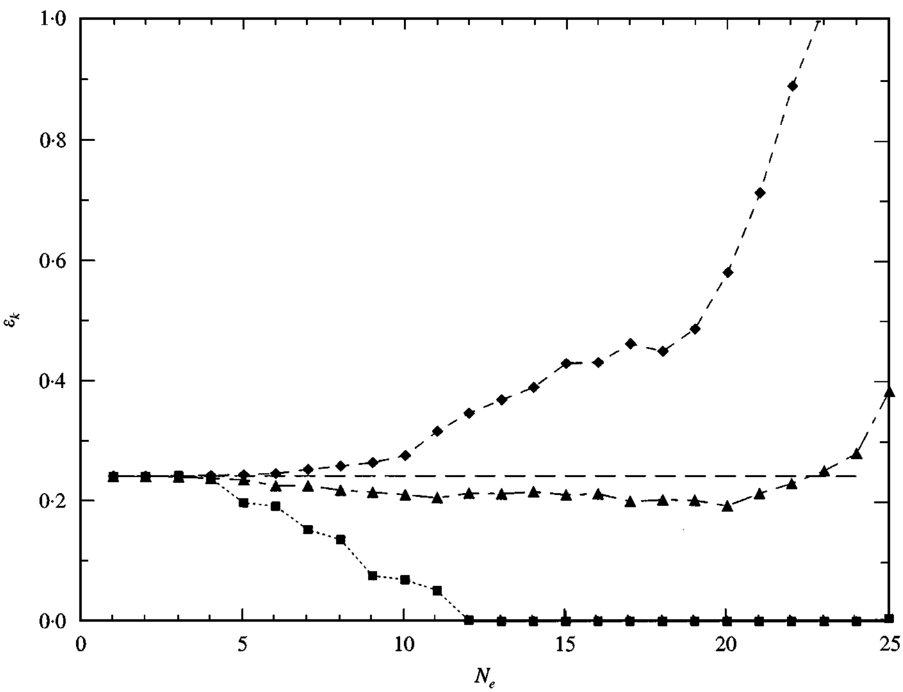

Figure 3. The stiffness error parameter, ε_k, obtained by using the proposed stiffness updating algorithm (■),the Lagrange multipliers scheme (▲), and the perturbation method (◆), as a function of the number of measured modes, N_e. The dotted horizontal line represents the stiffness error parameter of the analytical model, $(\varepsilon_k)_0$.

about the physical system, the more accurate the updated stiffnesses are. The Lagrange multipliers approach returns updated stiffnesses that are slightly better than the initial analytical stiffness values for N_e up to 22, after which it returns updated stiffnesses that are worst than the analytical ones. For $N_e \geqslant 2$, the perturbation approach returns updated stiffnesses that become increasingly worst as N_e increases.

Because updating the stiffness matrix of Figure 1 entails the solution of a least-square problem of size $N_e^2 \times (3N - 2)$ (see reference [3] for detailed discussion), to render the problem overdetermined requires $N_e \geqslant \sqrt{3N - 2}$ measured modes. Thus, for $N = 25$, at least nine measured modes are needed to update the stiffnesses to ensure sufficient accuracy based on the heuristics criterion. Because the criterion regarding the minimum N_e needed to perform the stiffness update is formulated empirically via numerical experiments, for the stiffness parameters of Table 1, 12 instead of nine measured modes are required to update the stiffnesses such that the resulting ε_k is nearly zero. As before, the numerical results indicate that there is a saturation point beyond which additional information does not lead to any significant improvement in the corrected stiffness matrix.

Depending on the set of system parameters, sometimes more and other times less measured modes than those predicted heuristically may be needed to perform the mass and stiffness updates. Nevertheless, using the empirical criteria regarding the smallest N_e needed to perform the update always leads to adjusted system matrices whose modes of vibration are substantially closer to the measured data than they were initially.

Figure 4 shows the resulting error parameters for the updated eigenvalues, ε_λ, as a function of N_e. The corresponding $(\varepsilon_\lambda)_0$ is also illustrated for comparison. Interestingly, while the perturbation updating schemes return updated mass and stiffness matrices that

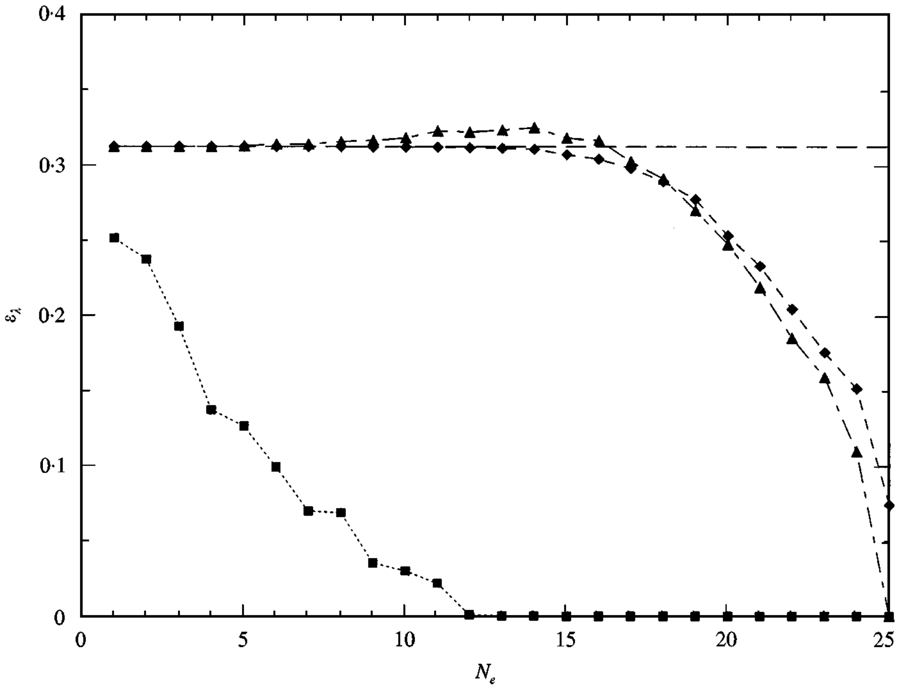

Figure 4. The eigenvalue error parameter, ε_λ, obtained by using the proposed mass/stiffness updating algorithms (■),, the Lagrange multipliers scheme (▲) and the perturbation method (◆), as a function of the number of measured modes, N_e. The dotted horizontal line represents the eigenvalue error parameter of the analytical model, $(\varepsilon_\lambda)_0$.

deviate substantially from those of the actual system, for a large set of measure modes, the resulting eigenvalues are close to those obtained during a modal survey. For $N_e \leqslant 13$, on the other hand, the eigenvalue error parameters for the perturbation approach are nearly identical to those of the analytical system, implying that the updated eigenvales are merely perturbations of the analytical ones. The eigenvalues obtained by using the Lagrange multipliers formalism track the measured eigenvalues well only when the set of measured modes is large. In fact, there exists a region ($6 \leqslant N_e \leqslant 16$) in which the Lagrange multipliers scheme returns updated eigenvalues that are *worst* than the initial analytical eigenvalues. As expected, because the measured eigensolutions are used as constraints in formulating the Lagrange multipliers updating algorithms, for $N_e = 25$, the Lagrange multipliers approach returns updated eigenvalues that are identical to those obtained during a modal survey. Consider now the updated eigenvalues obtained by using the proposed mass and stiffness correction schemes. Note how well the updated eigenvalues obtained by the proposed algorithms track the actual system, especially as N_e becomes large. In fact, for $N_e \geqslant 12$, the proposed model updating routines return updated mass and stiffness matrix that are nearly identical to the physical structure, and whose eigenvalues are nearly the same as those obtained during a modal survey.

The proposed mass and stiffness updating schemes require slightly more work and cause some down-time, because the modes of vibration for the mass-modified system need to be measured in order to correct the mass matrix of the structure. The additional time and effort, however, are a relatively small price to pay for the ability to update the analytical model so that its adjusted system matrices track those of the actual system accurately.

3. CONCLUSION

The effects of mode incompleteness on the quality of various mass and stiffness updating algorithms are studied. Specifically, the results of the newly developed updating schemes, based on using the modal surveys of the initial system and the corresponding mass-modified structure, are compared to those obtained by using the Lagrange multipliers formalism and the perturbation approach. The proposed updating schemes, which assume the connectivity information to be correct, are very for giving when it comes to deviations of the analytical system from the physical structure. Numerical experiments show that the proposed algorithms return updated mass and stiffness matrices that are more accurate than those obtained by using either the Lagrange multipliers approach or the perturbation method.

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